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Review text:

A family $\mathcal{F} \subseteq \binom{[n]}{k}$ is called *t-intersecting* if $|F \cap F'| \geq t$ for all $F, F' \in \mathcal{F}$, and two sets F, G are said to be *t-disjoint* if $|F \cap G| < t$. The structure of extremal t-intersecting families was completely determined by the theorem of Erdős–Ko–Rado [Quart. J. Math. Oxford Ser. (2) **12** (1961), 313–320; MR0140419] and the complete intersection theorem of Ahlswede–Khachatrian [European J. Combin. **18** (1997), no. 2, 125–136; MR1429238]. Two families $\mathcal{F}, \mathcal{G} \subseteq \binom{[n]}{k}$ are said to be *cross-t-intersecting* if $|F \cap G| \geq t$ for all $F \in \mathcal{F}$ and $G \in \mathcal{G}$, and there is a vast literature on determining the maximum product $|\mathcal{F}||\mathcal{G}|$ of sizes of such families; see Tokushige [J. Algebraic Combin. **38** (2013), no. 3, 653–662; MR3104733] for an eigenvalue method approach, and Zhang–Wu [J. Combin. Theory Ser. B **171** (2025), 49–70; MR4836581] for the resolution of a conjecture of Tokushige.

The authors consider the following relaxation: $\mathcal{F}$ and $\mathcal{G}$ are called *s-almost cross-t-intersecting* if each member of $\mathcal{F}$ (resp. $\mathcal{G}$) is t-disjoint with at most s members of $\mathcal{G}$ (resp. $\mathcal{F}$). The single-family analogue (i.e., s-almost t-intersecting families) has been studied by Gerbner–Lemons–Palmer–Patkós–Szécsi [SIAM J. Discrete Math. **26** (2012), no. 4, 1657–1669; MR3022158] and Frankl–Kupavskii [Electron. J. Combin. **28** (2021), no. 2, Paper No. 2.7, 16pp.; MR4245300], among others. For the cross version, Gerbner–Lemons–Palmer–Pálvölgyi–Patkós–Szécsi [Graphs Combin. **29** (2013), no. 3, 489–498; MR3053596] determined the maximum value of $|\mathcal{F}||\mathcal{G}|$ in the case $t = 1$.

In this paper, the authors find the maximum product $|\mathcal{F}||\mathcal{G}|$ for all $t \geq 1$. Write $\mathcal{H}_1([n], W; k) = \left\{H \in \binom{[n]}{k} : W \subseteq H\right\}$ for the full t-star centered at a t-set W. The authors show that for $k \geq t+1$ and $n \geq (t+1)(2(k-t+1)^2+7s)$, the maximum is attained if and only if $\mathcal{F} = \mathcal{G} = \mathcal{H}_1([n], W; k)$ for some $W \in \binom{[n]}{t}$. Note that this is the same extremal configuration as in the cross-t-intersecting case; the almost-intersection parameter s does not enlarge the maximum product.

The authors also characterize the extremal families among s-almost cross-t-intersecting families that are not cross-t-intersecting. For $k \geq t+2$, $|\mathcal{F}| \leq |\mathcal{G}|$, and n sufficiently large, they show that there exist $X \in \binom{[n]}{k+1}$ and $W \in \binom{X}{t}$ such that $\mathcal{F} = \mathcal{H}_1([n], W; k) \setminus \mathcal{A}$ and $\mathcal{G} = \mathcal{H}_1([n], W; k) \cup \mathcal{B}$, where $\mathcal{A}$ is a $\left(\binom{n-k-1}{k-t} - s\right)$-subset of $\left\{F \in \binom{[n]}{k} : F \cap X = W\right\}$ and $\mathcal{B}$ is a $\min\{t, s\}$-subset of $\binom{X}{k} \setminus \mathcal{H}_1(X, W; k)$. The case $k = t+1$ is a bit more delicate: when $t \geq s+2$, one family is a singleton $\{Y\}$ and the other is $\left\{G \in \binom{[n]}{t+1} : |G \cap Y| \geq t\right\}$ together with s additional sets; when $t \leq s+1$, the families are built around a $(t+s+2)$-set with a balanced covering property. By combining these results with a stability result for cross-t-intersecting families due to Cao–Lu–Lv–Wang [European J. Combin. **120** (2024), Paper No. 103958, 26 pp.; MR4728106], the authors obtain a complete characterization of the extremal s-almost cross-t-intersecting families subject to $\left|\bigcap_{H \in \mathcal{F} \cup \mathcal{G}} H\right| < t$; the answer depends on whether $k \geq 2t+1$ or $k \leq 2t$ (with the exception $(k,t) = (4,2)$).

The proofs are organized around case analysis on the pair of t-covering numbers $(\tau_t(\mathcal{F}), \tau_t(\mathcal{G}))$, where $\tau_t(\mathcal{F})$ denotes the minimum size of a subset $T \subseteq [n]$ satisfying $|T \cap F| \geq t$ for all $F \in \mathcal{F}$. A Bollobás-type set-pair inequality due to Oum–Wee [Combinatorica **38** (2018), no. 5, 1279–1284; MR3884790] is used to handle the cases where the covering numbers exceed k. The authors remark that the methods can also be applied to subspaces of a vector space over a finite field.